BOUNDS FOR THE MULTIPLICITY OF GORENSTEIN ALGEBRAS

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Abstract. We prove upper bounds for the Hilbert-Samuel multiplicity of standard graded Gorenstein algebras. The main tool that we use is Boij-Söderberg theory to obtain a decomposition of the Betti table of a Gorenstein algebra as the sum of rational multiples of symmetrized pure tables. Our bound agrees with the one in the quasi-pure case obtained by Srinivasan [J. Algebra, vol. 208, no. 2, (1998)].

1. Introduction

Let $\Bbbk$ be a field and R a standard graded polynomial ring over $\Bbbk$, i.e., $R = \bigoplus_{i \in \mathbb{N}} R_i$ as $\Bbbk$-vector-spaces, R_1 is a finite dimensional $\Bbbk$-vector-space and R is generated as a $\Bbbk$-algebra by R_1. Let M be a finitely generated graded R-module and $e(M)$ the Hilbert-Samuel multiplicity of M. We say that M is *Gorenstein* if M is Cohen-Macaulay and a minimal R-free resolution of M is self-dual. In this paper, we prove the following theorem:

Theorem 1.1. *Let M be a graded Gorenstein R-module that is minimally generated homogeneous elements of degree zero. Let $s = \operatorname{codim} M$ and $k = \lfloor \frac{s}{2} \rfloor$. Let $\beta_0(M)$ denote the minimal number of generators of M. For $0 \leq i \leq s$, write $m_i = m_i(M) = \min\{j : \operatorname{Tor}_i^R(\Bbbk, M)_j \neq 0\}$ and $M_i = M_i(M) = \max\{j : \operatorname{Tor}_i^R(\Bbbk, M)_j \neq 0\}$. Then*

$$e(M) \leq \frac{\beta_0(M)}{s!} \prod_{i=1}^{k} \min\left\{M_i, \left\lfloor \frac{m_s}{2} \right\rfloor\right\} \prod_{i=k+1}^{s} \max\left\{m_i, \left\lceil \frac{m_s}{2} \right\rceil\right\}.$$

There is a history of looking for bounds for $e(M)$ in terms of the homological invariants $m_i(M)$ and $M_i(M)$. Note that these are, respectively, the minimum and maximum twists in the free modules in a minimal R-free resolution of M. When $M = R/I$ for a homogeneous ideal I of R, then the conjectures of C. Huneke and Srinivasan, see [HS98, Conjecture 1], and of J. Herzog and Srinivasan, see [HS98, Conjecture 2], proposed bounds for $e(M)$. These conjectures were proved using the more general framework of Boij-Söderberg theory. The theory refers to the study of the decomposition of Betti tables of M generated graded R-modules in terms of extremal rays in the cone generated by the Betti tables of all the finitely generated R-modules, in a certain infinite dimensional rational vector space. This decomposition was conjectured by M. Boij and J. Söderberg [BS08, Conjecture 2.4], who showed that if this stronger conjecture were true, then the conjecture of Huneke and Srinivasan [HS98, Conjecture 1] and more would hold. This was in turn proved by D. Eisenbud, G. Fløystad and J. Weyman [EFW11] in characteristic zero and Eisenbud and F.-O. Schreyer [ES09] in a characteristic-free situation. Thereafter, Boij and Söderberg proved that the conjecture of Herzog and Srinivasan [HS98, Conjecture 2] holds. See [ES10] for a survey of Boij-Söderberg theory. We will need to use the Boij-Söderberg decomposition in our arguments; we have summarized the relevant features in Section 2.

When the resolution is quasi-pure, i.e. $m_i \geq M_{i-1}$ for all $i = 2, \cdots, s$, the aforementioned conjecture of Huneke and Srinivasan were proved using the equations of Peskine-Szpiro coming from the additivity of the Hilbert function [HS98, Theorem 1.2]. The duality of the resolution in the Gorenstein case can be exploited to obtain stronger bounds. In [Sri98, Theorem 4], Srinivasan showed that if R/I is Gorenstein and has a

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quasi-pure R-resolution, then

$$\frac{m_1 \cdots m_k M_{k+1} \cdots M_s}{s!} \le e(R/I) \le \frac{M_1 \cdots M_k m_{k+1} \cdots m_s}{s!}. \tag{1.2}$$

The notation here is the same as in Theorem 1.1: $s = \operatorname{ht} I$ and $k = \lfloor \frac{s}{2} \rfloor$. Theorem 1.1 generalizes the upper bound in this result to Gorenstein algebras with arbitrary resolutions.

Let $d = \dim M$. The *Hilbert coefficients* $e_i(M)$ with $0 \le i \le d$ are defined by expressing the Hilbert polynomial of M in the form

$$t \mapsto \sum_{i=1}^{d} (-1)^i e_i(M) \binom{t+d-1-i}{d-1-i}.$$

For each i, $e_i(M)$ is an integer and $e_0(M) = e(M)$. Using Boij-Söderberg theory, Herzog and X. Zheng obtained upper and lower bounds, in the fashion of [HS98, Conjecture 1], for all the Hilbert coefficients of arbitrary Cohen-Macaulay graded R-modules [HZ09, Theorem 2.1]. Extending [Sri98, Theorem 4], El Khoury and Srinivasan obtained upper and lower bounds for all Hilbert coefficients of Gorenstein quotient rings of R that have quasi-pure R-resolutions [ElS12, Theorem 4.2].

In this paper, we show that the duality in the minimal free resolution of a Gorenstein algebra can be captured in the Boij-Söderberg decomposition of its Betti table; this can be used to obtain stronger bounds for multiplicity. We define and use the notion of symmetrized pure Betti tables to capture the duality in the decomposition. Our upper bound recovers the upper bound in (1.2).

The computer algebra system `Macaulay2` [M2] provided valuable assistance in studying examples.

2. Preliminaries

Notation. As earlier, $\Bbbk$ is a field and $R = \Bbbk[x_1, \ldots, x_n]$ is a n-dimensional polynomial ring over $\Bbbk$ with $\deg x_i = 1$ for all $1 \le i \le n$. Let M be a finitely generated graded R-module. The *codimension* of M, denoted $\operatorname{codim} M$, is the codimension of the support of M in $\operatorname{Spec} R$. The *graded Betti numbers* of M are $\beta_{i,j}(M) = \dim_{\Bbbk} \operatorname{Tor}_i^R(\Bbbk, M)_j$. Note that $\beta_{i,j}(M)$ is the number of copies of $R(-j)$ that appear at homological degree i, in a minimal R-free resolution of M. We think of the collection $\{\beta_{i,j}(M) : 0 \le i \le n, j \in \mathbb{Z}\}$ as an element

$$\beta(M) = (\beta_{i,j}(M))_{0 \le i \le n, j \in \mathbb{N}} \in \mathbb{B} := \bigoplus_{i=0}^{n} \bigoplus_{j \in \mathbb{Z}} \mathbb{Q},$$

and call it the *Betti table* of M. In general, a *rational Betti table* β is an element $\beta = (\beta_{i,j})_{0 \le i \le n, j \in \mathbb{N}} \in \mathbb{B}$ such that:

(i) for all $0 \le i \le n$, $\beta_{i,j} = 0$ for all j such that $|j| \gg 0$,

(ii) for all $i > 0$ and for all j, if $\beta_{i,j} \neq 0$ then there exists $j' < j$ such that $\beta_{i-1,j'} \neq 0$.

Let $\beta = (\beta_{i,j})_{0 \le i \le n, j \in \mathbb{N}}$ be a rational Betti table. Its *length* is $\max\{i : \beta_{i,j} \neq 0 \text{ for some } j\}$. For $0 \le i \le \operatorname{length}(\beta)$, write $t_i(\beta) = \min\{j : \beta_{i,j} \neq 0\}$ and $T_i(\beta) = \max\{j : \beta_{i,j} \neq 0\}$.

Boij-Söderberg theory. We give here a summary of Boij-Söderberg theory that is relevant to us. For details, see [ES09, ES10]; an expository account is [Flø12].

A *degree sequence* of *length* s is an increasing sequence $d = (d_0 < d_1 < \cdots < d_s)$ of integers. For every such degree sequence d, there is a finitely generated graded Cohen-Macaulay R-module M of codimension s such that for all $0 \le i \le s$, $\beta_{i,j}(M) \neq 0$ if and only if $j = d_i$; for such a module M, we will say that d is the *type* of its resolution. Moreover, by the Herzog-Kühl equations [HK84], $\beta(M)$ is a positive rational multiple of the pure Betti table, which we denote by $\beta(d)$, given by:

$$\beta(d)_{i,j} = \begin{cases} \frac{1}{\prod_{l \neq i} |d_l - d_i|}, & 0 \le i \le s \text{ and } j = d_i \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $d = (d_0 < \cdots < d_s)$ be a degree sequence. We call the Betti table $\beta(d)$ defined in (2.1), the *pure Betti table* associated to d. For $0 \le i \le s$, write $\beta_i(d) = \beta(d)_{i,d_i}$. Eisenbud, Fløystad and Weyman [EFW11, Theorem 0.1] (in characteristic zero) and Eisenbud and Schreyer [ES09, Theorem 0.1] showed that for all

degree sequences d, there is a Cohen-Macaulay R-module M such that $\beta(M)$ is a rational multiple of $\beta(d)$. Moreover, for all R-modules M, $\beta(M)$ can be written as a non-negative rational combination of the $\beta(d)$; if we take a saturated chain of degree sequences, then the non-negative rational coefficients in the decomposition are unique [ES09, Theorem 0.2].

Self-dual resolutions and symmetrized Betti tables. Let β be a Betti table. Let s and N be integers. We say that β is (s,N)-*self-dual* if $\beta_{i,j} = \beta_{s-i,N-j}$ for all i,j. We say that β is *self-dual* if there exist s and N such that β is (s,N)-self-dual. If β is self-dual, then s is the length of β and $N = \max\{j : \beta_{s,j} \neq 0\} + \min\{j : \beta_{0,j} \neq 0\}$.

Definition 2.2. Let $d = (d_0 < \cdots < d_s)$ be a degree sequence and $N \geq d_0 + d_s$. Let $d^{\vee,N} = (N - d_s < \cdots < N - d_0)$. Denote the pure Betti table associated to $d^{\vee,N}$ by $\beta^{\vee,N}(d)$. Similarly, set $\beta_i^{\vee,N}(d) = \beta^{\vee,N}(d)_{i,N-d_{s-i}}$. Let $\beta_{\mathrm{sym}}(d,N) = \beta(d) + \beta^{\vee,N}(d)$. We call $\beta_{\mathrm{sym}}(d,N)$ the *symmetrized pure Betti table*, given by symmetrizing d with respect to N.

C. Peskine and L. Szpiro [PS74] observed that for a finitely generated graded R-module M with $s = \operatorname{codim} M$,

$$\sum_{i=0}^{\operatorname{pd} M} \sum_j (-1)^i \beta_{i,j} j^l = \begin{cases} 0 & \text{if } 0 \leq l < s \\ (-1)^s s! e(M) & \text{if } l = s. \end{cases}$$

(Here $\operatorname{pd} M$ is the projective dimension of M, or equivalently, the length of $\beta(M)$.) Suppose that d is a degree sequence of length s. Since the pure Betti table $\beta(d)$ is, up to multiplication by a rational number, the Betti table of a Cohen-Macaulay R-module of codimension s, we see that $\sum_{i=0}^{s}(-1)^i \beta_i(d) d_i^l = 0$ for all $0 \leq l < s$. Further, by direct calculation using (2.1) we can see that$\sum_{i=0}^{s}(-1)^i \beta_i(d) d_i^s = (-1)^s$. Therefore we set

$$e(\beta(d)) = \frac{1}{s!} \quad \text{and} \quad e(\beta_{\mathrm{sym}}(d,N)) = \frac{2}{s!}. \tag{2.3}$$

We now argue that the Betti table of a Gorenstein module can be decomposed into a non-negative rational combination of symmetrized pure Betti tables.

Proposition 2.4. *Let M be finitely generated graded Cohen-Macaulay R-module with* $\operatorname{codim} M = s$, *generated minimally by homogeneous elements of degree zero. Suppose that $\beta(M)$ is self-dual. Let $N = T_s(M)$. Then there exist degree sequences $d^\alpha, 0 \leq \alpha \leq a$ for some $a \in \mathbb{N}$ and positive rational numbers $r_\alpha, 0 \leq \alpha \leq a$ such that*

$$\beta(M) = \sum_{\alpha=0}^{a} r_\alpha \beta_{\mathrm{sym}}(d^\alpha, N).$$

Moreover,

(i) the d^α are degree sequences of length s and they are not (s,N)-dual to each other.
(ii) $d^{\alpha+1} > d^\alpha$ for all $0 \leq \alpha \leq a-1$.
(iii) $N \geq d_i^\alpha + d_{s-i}^\alpha$ for all α and i, or equivalently, $d^\alpha \leq (d^\alpha)^{\vee,N}$ for all α.

Proof. We prove this similar to the Decomposition Algorithm of [ES09, p. 864]. First, for the duration of this proof, we will say that a rational Betti table $\beta \geq 0$ if $\beta_{i,j} \geq 0$ for all i,j. Let $d^0 = (0 < t_1(M) < \cdots < t_s(M) = N)$. Then there exist positive rational numbers r_0 and r_0' such that $\beta(M) - r_0\beta(d^0) + r_0'\beta((d^0)^{\vee,N}) \geq 0$.

Let $d^\alpha, 0 \leq \alpha \leq a$ be a maximal (by inclusion) saturated chain of degree sequences with $d_0^\alpha = 0$ and $d_s^\alpha = N$ for all α such that $(d^\alpha)^{\vee,N} \geq d^\alpha$. If we repeat this procedure, we see that there exist non-negative rational numbers r_α and r_α' for all $0 \leq \alpha \leq a$, that are uniquely determined, such that

$$\beta(M) = \sum_{\alpha=0}^{a} r_\alpha \beta(d^\alpha) + \sum_{\alpha=0}^{a} r_\alpha' \beta((d^\alpha)^{\vee,N}).$$

Since $\beta(M)$ is (s,N)-self-dual, $r_\alpha = r_\alpha'$ for all α and all the assertions follow immediately. ∎

3. Main Theorem

In this section, we prove Theorem 1.1, which we restate here for the sake of convenience.

Theorem 3.1. *Let M be a graded Gorenstein R-module that is minimally generated homogeneous elements of degree zero. Let $s = \operatorname{codim} M$ and $k = \lfloor \frac{s}{2} \rfloor$. Let $\beta_0(M)$ denote the minimal number of generators of M. For $0 \le i \le s$, write $m_i = m_i(M) = \min\{j : \operatorname{Tor}_i^R(\Bbbk, M)_j \neq 0\}$ and $M_i = M_i(M) = \max\{j : \operatorname{Tor}_i^R(\Bbbk, M)_j \neq 0\}$. Then*

$$e(M) \le \frac{\beta_0(M)}{s!} \prod_{i=1}^{k} \min\left\{M_i, \left\lfloor \frac{m_s}{2} \right\rfloor\right\} \prod_{i=k+1}^{s} \max\left\{m_i, \left\lceil \frac{m_s}{2} \right\rceil\right\}.$$

Definition 3.2. Let $d = (0, d_1, \dots, d_s)$ be a degree sequence such that $d_s \ge d_i + d_{s-i}$ for all $0 \le i \le s$. Let $b_d = \beta_0(d) + \beta_0(d^{\vee, d_s})$ and

$$\Psi_d = \prod_{i=1}^{k} \min\left\{d_s - d_{s-i}, \left\lfloor \frac{d_s}{2} \right\rfloor\right\} \prod_{i=k+1}^{s} \max\left\{d_i, \left\lceil \frac{d_s}{2} \right\rceil\right\}.$$

For two degree sequences $d = (d_0 < \cdots < d_s)$ and $d' = (d'_0 < \ldots < d'_s)$, we say that $d < d'$ if $d_i \le d'_i$ for all $0 \le i \le s$ and $d \neq d'$.

Lemma 3.3. *Let d and d' be degree sequences such that $d_0 = 0$ and $d < d' \le (d')^{\vee, d_s} < d^{\vee, d_s}$. Then $\Psi_d > \Psi_{d'}$.*

Proof. By induction on $\sum_i d'_i - d_i$, we may assume, without loss of generality, that there exists j such that $d'_j = d_j + 1$ and $d'_i = d_i$ for all $i \neq j$. Moreover, if $1 \le i \le s - k - 1$, then d_i does not figure in the expression for Ψ_d, so we may assume that $j \ge s - k$. Additionally, $j \le s - 1$. Let us rewrite Ψ_d as

$$\Psi_d = \prod_{i=s-k}^{s-1} \min\left\{d_s - d_i, \left\lfloor \frac{d_s}{2} \right\rfloor\right\} \prod_{i=k+1}^{s} \max\left\{d_i, \left\lceil \frac{d_s}{2} \right\rceil\right\}. \tag{3.4}$$

Two cases arise: $j < k + 1$ and $j \ge k + 1$. The first case is possible if and only if $s = 2k$ and $j = k$. In this case, d_k appears only once in (3.4), and since $d_k \le d_s - d_{s-k} = d_s - d_k$, we get $d_s - d_k \ge \lfloor \frac{d_s}{2} \rfloor$. By the hypothesis that $d' \le (d')^{\vee, d_s}$, $d_s - d_k - 1 \ge d_k + 1$, so $d_s - d_k - 1 \ge \lfloor \frac{d_s}{2} \rfloor$. Hence

$$\frac{\Psi_{d'}}{\Psi_d} = \frac{d_s - d_k - 1}{d_s - d_k} \le 1.$$

In the second case (i.e., $j \ge k + 1$), d_j appears twice in in (3.4). However, note that

$$\max\left\{d_j + 1, \left\lceil \frac{d_s}{2} \right\rceil\right\} \ge \max\left\{d_j, \left\lceil \frac{d_s}{2} \right\rceil\right\} \ge \left\lceil \frac{d_s}{2} \right\rceil, \qquad \text{and,}$$
$$\min\left\{d_s - d_j - 1, \left\lfloor \frac{d_s}{2} \right\rfloor\right\} \le \min\left\{d_s - d_j, \left\lfloor \frac{d_s}{2} \right\rfloor\right\} \le \left\lfloor \frac{d_s}{2} \right\rfloor$$

so

$$\Psi_{d'} = \Psi_d \frac{\max\{d_j + 1, \lceil \frac{d_s}{2} \rceil\} \min\{d_s - d_j - 1, \lfloor \frac{d_s}{2} \rfloor\}}{\max\{d_j, \lceil \frac{d_s}{2} \rceil\} \min\{d_s - d_j, \lfloor \frac{d_s}{2} \rfloor\}} \le \Psi_d. \qquad \blacksquare$$

The following proposition shows that Theorem 1.1 holds for symmetrized pure Betti tables.

Proposition 3.5. *Let $d = (0, d_1, \dots, d_s)$ be a degree sequence such that $d \le d^{\vee, d_s}$. Then $b_d \Psi_d \ge 2$.*

Proof. We will prove this by induction on $\sum_i (d_s - d_{s-i} - d_i)$, which is non-negative by our hypothesis. If $\sum_i (d_s - d_{s-i} - d_i) = 0$ (equivalently, $d = d^{\vee, d_s}$), then the assertion is true. If $d < d^{\vee, d_s}$, then there exists $j > k$ such that $d_j < d_s - d_{s-j}$. Pick j to be maximal with this property. Let $d' = (0, d_1, \cdots, d_{j-1}, d_j + 1, d_{j+1}, \cdots, d_s)$. Then $d' \le (d')^{\vee}$ and $\sum_i (d_s - d_{s-i} - d_i) > \sum_i (d'_s - d'_{s-i} - d'_i)$, so by induction, $b_{d'} \Psi_{d'} \ge \frac{2}{s!}$.

We now show that $b_d\Psi_d \geq b_{d'}\Psi_{d'}$. If $b_d \geq b_{d'}$, then it is true by Lemma 3.3. Hence suppose that $b_d < b_{d'}$. In particular $d_j > \frac{d_s}{2}$, and hence $j > k$. Therefore

$$\frac{\Psi_{d'}}{\Psi_d} = \frac{(d_s - d_j - 1)(d_j + 1)}{(d_s - d_j)d_j}.$$

Therefore it suffices to show that $b_{d'}(d_s - d_j - 1)(d_j + 1) \leq b_d(d_s - d_j)d_j$. Let

$$\xi_1 = \prod_{\substack{i=0\\ i\neq j}}^{s-1} \frac{1}{d_s - d_i} \quad \text{and} \quad \xi_2 = \prod_{\substack{i=1\\ i\neq j}}^{s} \frac{1}{d_i}.$$

Then $b_{d'}(d_s - d_j - 1)(d_j + 1) = (d_j + 1)\xi_1 + (d_s - d_j - 1)\xi_2$ and $b_d(d_s - d_j)d_j = d_j\xi_1 + (d_s - d_j)\xi_2$. Then $b_d(d_s - d_j)d_j - b_{d'}(d_s - d_j - 1)(d_j + 1) = \xi_2 - \xi_1$. We can see that $\xi_2 - \xi_1 \geq 0$ by noting that, for all i, the ith element of the sequence $(d_s - d_{s-1}) < \cdots < (d_s - d_{j+1}) < (d_s - d_{j-1}) < \cdots < (d_s - d_1) < d_s$ is at least as large as the ith element of $d_1 < \cdots < d_{j-1} < d_{j+1} < \cdots < d_s$, since $j > k$. ■

Proof of Theorem 1.1. Pick degree sequences d^α and non-negative rational numbers r_α as in Proposition 2.4. We need to show that $s!e(M) \leq \beta_0(M)\Psi_{d^0}$. We get this as follows: $s!e(M) = s!\sum_\alpha r_\alpha e(\beta_{sym}(d^\alpha)) = \sum_\alpha 2r_\alpha \leq \sum_\alpha r_\alpha b_{d^\alpha}\Psi_{d^\alpha} \leq \left(\sum_\alpha r_\alpha b_{d^\alpha}\right)\Psi_{d^0} = \beta_0(M)\Psi_{d^0}$, where the two equalities follow from Proposition 2.4 and (2.3), and the two inequalities follow from Proposition 3.5 and from Lemma 3.3, respectively. Since $d^0 = (0, t_1, t_2, \ldots, t_s) = (1, m_1, \ldots, m_s = d_s)$, we get the result. ■

Lower bounds. We have not been able to find an analogous generalization of the lower bound in [Sri98, Theorem 4] (see (1.2)) to the non-quasi-pure case. As an example, consider $R = k[x, y, z]$ and $I = (yz, xz, xy, y^7 - z^7, x^7 - z^7)$. Note that $e(R/I) = 20$. The minimal free resolution of R/I is:

$$0 \longrightarrow R(-10) \longrightarrow R(-8)^3 \oplus R(-3)^2 \longrightarrow R(-7)^2 \oplus R(-2)^3 \longrightarrow R \longrightarrow 0$$

Indeed $e(R/I) \not\geq \frac{m_1M_2M_3}{6} = \frac{160}{6}$. It will be interesting to obtain strong lower bounds.

Special bounds for codimension 3. Suppose that $R = \Bbbk[x, y, z]$ and I is a homogeneous (x, y, z)-primary R-ideal such that R/I is Gorenstein. Write $N_1 = \max\{j : \beta_{1,j}(R/I) > \beta_{2,j}(R/I)\}$. Migliore, Nagel and Zanello show that (see [MNZ08, Theorem 3.1])

$$e(R/I) \leq \frac{N_1T_2d_3}{6}. \tag{3.6}$$

The bound in (3.6) is not comparable with that from Theorem 1.1. We give examples to show this. To begin with, note that, in the codimension-three situation, the bound from Theorem 1.1 can be rewritten as

$$e(R/I) \leq \begin{cases} \frac{M_1m_2m_3}{6}, & \text{if } R/I \text{ has a quasi-pure resolution, i.e., } M_1 \leq m_2, \\ \frac{\lfloor\frac{m_3}{2}\rfloor\lceil\frac{m_3}{2}\rceil m_3}{6}, & \text{otherwise.} \end{cases} \tag{3.7}$$

Suppose that R/I has a quasi-pure resolution. If the degree of the socle of R/I is even, or equivalently, m_3 is odd, then $M_1 = m_2 - 1$. Hence $\beta_{1,M_1} > 0 = \beta_{2,M_1}$, so $N_1 = M_1$. In this case, the bound from (3.7) is strictly smaller than the bound from (3.6). On the other hand, if the socle lives in an odd degree, then $M_1 = m_2$ and $\beta_{1,M_1} = \beta_{2,M_1}$, so $N_1 = \max\{j < M_1 : \beta_{1,j} \neq 0\}$. From the next two examples, we see that neither bound performs better than the other in this situation. Consider, first, $I = (x^2, y^2, z^4)$ in $\Bbbk[x, y, z]$. Since $N_1 = 2$, the bound $\frac{2\cdot6\cdot8}{6} = 16$ of (3.6) is smaller than the bound $\frac{4\cdot4\cdot8}{6} = 21\frac{1}{3}$ of (3.7). Now consider the 4×4 Pfaffians of a skew-symmetric map from $R(-6) \oplus R(-7) \oplus R(-7) \oplus R(-8) \oplus R(-8)$ to $R(-4) \oplus R(-4) \oplus R(-5) \oplus R(-5) \oplus R(-6)$, constructed using the following `Macaulay2` [M2] code.

```
R = QQ[x,y,z];
random(R^{-4, -4, -5, -5, -6}, R^{-8, -8, -7, -7, -6});
phi = oo-transpose(oo);
I = pfaffians(4,phi);
betti res I
```

For this example, $N_1 = 5$. Hence the bounds in (3.7) and (3.6), respectively, are $\frac{5\cdot 8\cdot 12}{6} = 80$ and $\frac{6\cdot 6\cdot 12}{6} = 72$.

The next two examples show that similar behaviour can be expected in the case non-quasi-pure resolutions. If $I = (yz, xz, x^3+x^2y-xy^2-2y^3, x^2y^2-y^4, xy^4-z^5)$, then the bound of (3.6) is $\frac{2\cdot 6\cdot 8}{6} = 16$ and that of (3.7) is $\frac{4\cdot 4\cdot 8}{6} = 21\frac{1}{3}$. If $I = (yz, xz, x^2y^2-xy^3+y^4, x^4+x^3y+2xy^3, y^6-z^6)$, then the bounds are $\frac{6\cdot 7\cdot 9}{6} = 63$ (3.6) and $\frac{4\cdot 5\cdot 9}{6} = 30$ (3.7).

References

[BS08] M. Boij and J. Söderberg, *Graded Betti numbers of Cohen-Macaulay modules and the multiplicity conjecture*, J. Lond. Math. Soc. (2) **78** (2008), no. 1, 85–106. 1

[EFW11] D. Eisenbud, G. Fløystad, and J. Weyman, *The existence of equivariant pure free resolutions*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 3, 905–926. 1, 2

[ES09] D. Eisenbud and F. -O. Schreyer, *Betti numbers of graded modules and cohomology of vector bundles*, J. Amer. Math. Soc. **22** (2009), no. 3, 859–888. 1, 2, 3

[ES10] ———, *Betti numbers of syzygies and cohomology of coherent sheaves*, Proceedings of the International Congress of Mathematicians, 2010, Hyderabad, India, pp. 1–16. 1, 2

[ElS12] S. El Khoury and H. Srinivasan *Gorenstein Hilbert Coefficients*, arXiv:1201.5575v3, 2012. 2

[Flø12] G. Fløystad, *Boij-Söderberg theory: introduction and survey*, Progress in commutative algebra 1, de Gruyter, Berlin, 2012, pp. 1–54. 2

[HK84] J. Herzog and M. Kühl, *On the Betti numbers of finite pure and linear resolutions*, Comm. Algebra **12** (1984), no. 13-14, 1627–1646. 2

[HS98] J. Herzog and H. Srinivasan, *Bounds for multiplicities*, Trans. Amer. Math. Soc. **350** (1998), no. 7, 2879–2902. 1, 2

[HZ09] J. Herzog and X. Zheng, *Bounds for Hilbert coefficients*, Proc. Amer. Math. Soc. **137** (2009), no. 2, 487–494. 2

[M2] D. R. Grayson, M. E. Stillman, Macaulay 2, a software system for research in algebraic geometry, Available at http://www.math.uiuc.edu/Macaulay2/. 2, 5

[MNZ08] J. Migliore, U. Nagel, and F. Zanello, *An improved multiplicity conjecture for codimension 3 Gorenstein algebras*, Comm. Algebra **36** (2008), no. 1, 112–119. 5

[PS74] C. Peskine and L. Szpiro, *Syzygies et multiplicités*, C. R. Acad. Sci. Paris Sér. A **278** (1974), 1421–1424. 3

[Sri98] H. Srinivasan, *A note on the multiplicities of Gorenstein algebras*, J. Algebra **208** (1998), no. 2, 425–443. 1, 2, 5

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